

Semisimple Triangular Hopf Algebras and Tannakian Categories

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Abstract

In this paper we provide a complete and detailed proof of Theorem 2.1 from [EG1]. This theorem states that any semisimple triangular Hopf algebra over an algebraically closed field k of characteristic 0 is obtained from a group algebra $k[G]$, of a unique (up to isomorphism) finite group G , by twisting its usual comultiplication in the sense of Drinfeld [Dr]. This result is one of the key theorems in the classification of semisimple and cosemisimple triangular Hopf algebras over *any* algebraically closed field, which was obtained in [EG4]. The proof of this theorem relies on Deligne's theorem on Tannakian categories [De] in an essential way. We thus, in particular, review the notions of tensor, rigid, symmetric and Tannakian categories, discuss Deligne's theorem, and study the category of finite-dimensional representations of a finite-dimensional triangular (semisimple) Hopf algebra.

1 Introduction

One of the most fundamental problems in the theory of Hopf algebras is the classification and construction of finite-dimensional Hopf algebras over an algebraically closed field k of characteristic 0. However, this problem is so difficult that until today there exists in the literature only a single general classification result; namely, the classification of Hopf algebras of prime dimension. Kaplansky conjectured in 1975 that any prime dimensional Hopf algebra over k is isomorphic to the group algebra $k[\mathbb{Z}_p]$ [Kap], and it was only in 1994 that this conjecture was settled by Zhu [Z]. Therefore people restrict the problem to certain classes of finite-dimensional Hopf algebras, e.g. to semisimple ones (these are always finite-dimensional by a theorem of Sweedler [S]). In many senses semisimple Hopf algebras

over k deserve to be considered as "quantum" analogue of finite groups, but even so, the problem remains extremely hard even in low dimensions. So far, proceeding the classification of semisimple Hopf algebras over k by the dimension has not proved to be very fruitful. In fact, except for a very few special low dimensions (see e.g. [Mon2] and references therein), the only general classification theorems known today in the literature are in dimension pq and p^3 where p, q are prime numbers. In the first case it was proved that semisimple Hopf algebras of dimension pq are either group algebras or duals of group algebras [EG7], and in the second case Masuoka proved that there are exactly $p + 1$ non-isomorphic semisimple Hopf algebras of dimension p^3 which are neither group algebras nor duals of group algebras.

A great boost to the theory of (finite-dimensional) Hopf algebras was given by Drinfeld in the mid 80's when he invented the so called *quasitriangular* Hopf algebras [Dr] for the purpose of constructing solutions to the quantum Yang-Baxter equation that arises in mathematical physics. Quasitriangular Hopf algebras are the Hopf algebras whose category of finite-dimensional representations is *braided rigid*, and thus they have intriguing relationships also with low dimensional topology. A quasitriangular Hopf algebra is called *triangular* if and only if the corresponding braided rigid category is *symmetric*; just like the categories of finite-dimensional representations of a group or a Lie algebra are. In particular, Drinfeld showed that *any* finite-dimensional Hopf algebra can be embedded into a finite-dimensional quasitriangular Hopf algebra known as its Drinfeld (or quantum) double. It is thus natural to focus on the problem of classification and construction of semisimple (quasi)triangular Hopf algebras. For semisimple quasitriangular Hopf algebras the problem is still widely open. However, the theory of semisimple triangular Hopf algebras over k is essentially closed now [EG1-EG4].

The purpose of this paper is to explain in full details how Deligne's theorem on Tannakian categories [De] was applied in [EG1] to prove the key structure theorem about semisimple triangular Hopf algebras over k . This theorem states that any such Hopf algebra is obtained from a group algebra $k[G]$ of a unique (up to isomorphism) finite group G by twisting its usual comultiplication in the sense of Drinfeld (see Theorem 6.1 below). This result is the key theorem in the classification of semisimple triangular Hopf algebras over k which was completed in [EG4] (see also [G2]).

The paper is organized as follows. In Section 2 we recall the definition of a symmetric rigid category and some of the fundamental examples of such categories. In Section 3 we recall the definition of a Tannakian category, and recall a Theorem of Deligne and Milne [DM] on the structure of such categories, and a Theorem of Deligne [De] stating an intrinsic characterization of such categories in terms of the categorical dimensions of objects. In Section 4 we turn to Hopf algebras and prove that the category of finite-dimensional representations of a triangular Hopf algebra is symmetric and rigid. In Section 5 we focus on a special class of triangular Hopf algebras obtained from group algebras of finite groups by twisting their usual comultiplications. In Section 6 we explain how Deligne's theorem is applied in order to show that these are in fact all the semisimple triangular Hopf algebras over k (see Theorem 6.1 below). In Section 7 we conclude the paper with brief discussions of some generalizations and applications of Theorem 6.1, and some other related topics.

We refer the reader to the books [ES,Kas,Mon1,S] as references for the general theory of Hopf algebras and quantum groups.

Throughout this paper, unless otherwise specified, the ground field k is assumed to be algebraically closed of characteristic 0, and all categories are assumed to be k -linear and abelian.

2 Symmetric rigid categories

In this section we recall the definition of a symmetric rigid category and some basic examples of such categories. The notion of a symmetric rigid category, introduced by Mac Lane [Mac1], is an abstract formulation of the properties of the category of finite-dimensional vector spaces over a field.

Definition 2.1 *A tensor category is a sextuple $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ where \mathcal{C} is a category, $\mathbf{1} \in \text{Ob}(\mathcal{C})$ is the unit object, \otimes is a functor*

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}, \quad (1)$$

a is a natural isomorphism (associativity constraint)

$$a : \otimes \circ (id_{\mathcal{C}} \times \otimes) \rightarrow \otimes \circ (\otimes \times id_{\mathcal{C}}) \quad (2)$$

such that the diagram

$$\begin{array}{ccc} W \otimes (X \otimes (Y \otimes Z)) & \xrightarrow{a_{W,X,Y \otimes Z}} & (W \otimes X) \otimes (Y \otimes Z) \xrightarrow{a_{W \otimes X,Y,Z}} ((W \otimes X) \otimes Y) \otimes Z \\ id_W \otimes a_{X,Y,Z} \downarrow & & a_{W,X,Y} \otimes id_Z \uparrow \\ W \otimes ((X \otimes Y) \otimes Z) & \xrightarrow{a_{W,X \otimes Y,Z}} & (W \otimes (X \otimes Y)) \otimes Z \end{array} \quad (3)$$

is commutative for all $W, X, Y, Z \in \text{Ob}(\mathcal{C})$ (the Pentagon Axiom), and l_X, r_X are natural isomorphisms (unit constraints)

$$l_X : \mathbf{1} \otimes X \rightarrow X, \quad r_X : X \otimes \mathbf{1} \rightarrow X \quad (4)$$

such that the diagram

$$\begin{array}{ccc} X \otimes (\mathbf{1} \otimes Y) & \xrightarrow{a_{X,\mathbf{1},Y}} & (X \otimes \mathbf{1}) \otimes Y \\ id_X \otimes l_Y \downarrow & & r_X \otimes id_Y \downarrow \\ X \otimes Y & \xrightarrow{id_{X \otimes Y}} & X \otimes Y \end{array} \quad (5)$$

is commutative for all $X, Y \in \text{Ob}(\mathcal{C})$.

Remark 2.2 The meaning of (3) and (5) is the following. Let $X_1, \dots, X_n \in \text{Ob}(\mathcal{C})$. Then by Mac Lane's Coherence Theorem [Mac2], any two expressions obtained from $X_1 \otimes \dots \otimes X_n$ by inserting $\mathbf{1}$ and brackets are isomorphic via an isomorphism composed only from the constraints l, r, a and their inverses.

There is a natural notion of morphism between two tensor categories.

Definition 2.3 Let $\tilde{\mathcal{C}} := (\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ and $\tilde{\mathcal{C}}' := (\mathcal{C}', \otimes', \mathbf{1}', a', l', r')$ be two tensor categories. A tensor functor from $\tilde{\mathcal{C}}$ to $\tilde{\mathcal{C}}'$ is a triple (F, J, φ) where $F : \mathcal{C} \rightarrow \mathcal{C}'$ is a k -linear functor, $J = \{J_{X,Y} : F(X) \otimes' F(Y) \rightarrow F(X \otimes Y) | X, Y \in \text{Ob}(\mathcal{C})\}$ is a functorial isomorphism, and $\varphi : \mathbf{1}' \rightarrow F(\mathbf{1})$ is an isomorphism, such that the diagrams

$$\begin{array}{ccc}
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{a'_{F(X), F(Y), F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\
J_{X,Y} \otimes' id_{F(Z)} \downarrow & & id_{F(X)} \otimes' J_{Y,Z} \downarrow \\
F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\
J_{X \otimes Y, Z} \downarrow & & J_{X, Y \otimes Z} \downarrow \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X \otimes (Y \otimes Z))
\end{array} \tag{6}$$

$$\begin{array}{ccc}
\mathbf{1}' \otimes' F(X) & \xrightarrow{l'_{F(X)}} & F(X) \\
\varphi \otimes' id_{F(X)} \downarrow & & F(l_X) \uparrow \\
F(\mathbf{1}) \otimes' F(X) & \xrightarrow{J_{\mathbf{1}, X}} & F(\mathbf{1} \otimes X)
\end{array} \tag{7}$$

and

$$\begin{array}{ccc}
F(X) \otimes' \mathbf{1}' & \xrightarrow{r'_{F(X)}} & F(X) \\
id_{F(X)} \otimes' \varphi \downarrow & & F(r_X) \uparrow \\
F(X) \otimes' F(\mathbf{1}) & \xrightarrow{J_{X, \mathbf{1}}} & F(X \otimes \mathbf{1})
\end{array} \tag{8}$$

are commutative for all $X, Y, Z \in \text{Ob}(\mathcal{C})$.

There is a natural notion of morphism between two tensor functors.

Definition 2.4 Let $\tilde{\mathcal{C}} := (\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ and $\tilde{\mathcal{C}}' := (\mathcal{C}', \otimes', \mathbf{1}', a', l', r')$ be two tensor categories, and $(F_1, J_1, \varphi_1), (F_2, J_2, \varphi_2)$ two tensor functors from $\tilde{\mathcal{C}}$ to $\tilde{\mathcal{C}}'$. A morphism of tensor functors $\eta : (F_1, J_1, \varphi_1) \rightarrow (F_2, J_2, \varphi_2)$ is a natural transformation $\eta : F_1 \rightarrow F_2$ such that $\eta_{\mathbf{1}} \circ \varphi_1 = \varphi_2$, and the diagram

$$\begin{array}{ccc}
F_1(X) \otimes' F_1(Y) & \xrightarrow{J_1} & F_1(X \otimes Y) \\
\eta_X \otimes' \eta_Y \downarrow & & \eta_{X \otimes Y} \downarrow \\
F_2(X) \otimes' F_2(Y) & \xrightarrow{J_2} & F_2(X \otimes Y)
\end{array} \tag{9}$$

is commutative for all $X, Y \in \text{Ob}(\mathcal{C})$. The notions of isomorphism and automorphism, and that of an (anti-)equivalence between two tensor categories are defined accordingly.

The following are basic examples of tensor categories.

Example 2.5 Let k be any field.

1. The category Vec_k of all finite-dimensional vector spaces over k is a tensor category, where $\otimes = \otimes_k$, $\mathbf{1} = k$, and the constraints a, l, r are the usual ones.

2. Let G be an affine (pro)algebraic group. The category $\text{Rep}_k(G)$ of all finite-dimensional algebraic representations of G over k is a k -linear abelian tensor category. Note that the forgetful functor $\text{Forget} : \text{Rep}_k(G) \rightarrow \text{Vec}_k$ has a tensor structure $(\text{Forget}, J, \varphi)$ where $J_{X,Y} = id_{X \otimes Y}$ for all $X, Y \in \text{Ob}(\text{Rep}_k(G))$, and $\varphi = id_k$. In fact, G can be reconstructed from $\text{Rep}_k(G)$ and $(\text{Forget}, J, \varphi)$. Indeed, $G \cong \text{Aut}^\otimes(\text{Forget})$ the group of tensor automorphisms of Forget (see e.g. [ES, 18.2.2]).

Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a tensor category, and $X \in \text{Ob}(\mathcal{C})$. An object $X^* \in \text{Ob}(\mathcal{C})$ is said to be a (left) dual of X if there exist two morphisms $ev_X : X^* \otimes X \rightarrow \mathbf{1}$ and $coev_X : \mathbf{1} \rightarrow X \otimes X^*$ such that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{coev_X \otimes id_X} & (X \otimes X^*) \otimes X \\ id_X \uparrow & & a_{X, X^*, X} \uparrow \\ X & \xleftarrow{id_X \otimes ev_X} & X \otimes (X^* \otimes X) \end{array} \quad (10)$$

and

$$\begin{array}{ccc} X^* & \xrightarrow{id_{X^*} \otimes coev_X} & X^* \otimes (X \otimes X^*) \\ id_{X^*} \downarrow & & a_{X^*, X, X^*} \downarrow \\ X^* & \xleftarrow{ev_X \otimes id_{X^*}} & (X^* \otimes X) \otimes X^* \end{array} \quad (11)$$

are commutative. If $X \in \text{Ob}(\mathcal{C})$ has a dual object, then it is unique up to isomorphism. If any object of \mathcal{C} has a dual object, then one can define the (contravariant) dual object functor $\mathcal{C} \rightarrow \mathcal{C}$ as follows: for $X, Y \in \text{Ob}(\mathcal{C})$ and $f : X \rightarrow Y$, $X \mapsto X^*$ and $f \mapsto f^*$ where $f^* : Y^* \rightarrow X^*$ is defined by

$$f^* := l_{X^*} \circ (ev_Y \otimes id_{X^*}) \circ (id_{Y^*} \otimes f \otimes id_{X^*}) \circ a_{Y^*, X, X^*} \circ (id_{Y^*} \otimes coev_X) \circ (r_{Y^*})^{-1}.$$

Definition 2.6 A tensor category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ is called rigid if every object $X \in \text{Ob}(\mathcal{C})$ has a dual object, and the dual object functor is an anti-equivalence of tensor categories.

Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a tensor category, and $\tau : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ be the permutation functor.

Definition 2.7 A symmetric category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$ is a tensor category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ equipped with a natural isomorphism $c : \otimes \rightarrow \otimes \circ \tau$ (commutativity constraint) such that the diagram

$$\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{id_X \otimes c_{Y,Z}} & X \otimes (Z \otimes Y) \\
a_{X,Y,Z} \downarrow & & a_{X,Z,Y} \downarrow \\
(X \otimes Y) \otimes Z & & (X \otimes Z) \otimes Y \\
c_{X \otimes Y, Z} \downarrow & & c_{X,Z} \otimes id_Y \downarrow \\
Z \otimes (X \otimes Y) & \xrightarrow{a_{Z,X,Y}} & (Z \otimes X) \otimes Y
\end{array} \tag{12}$$

is commutative for all $X, Y, Z \in \text{Ob}(\mathcal{C})$ (Hexagon Axiom), and

$$c_{X,Y} \circ c_{Y,X} = id_{Y \otimes X} \tag{13}$$

for all $X, Y \in \text{Ob}(\mathcal{C})$.

There is a natural notion of morphism between two symmetric categories.

Definition 2.8 Let $\tilde{\mathcal{C}} := (\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$ and $\tilde{\mathcal{C}}' := (\mathcal{C}', \otimes', \mathbf{1}', a', l', r', c')$ be two symmetric categories. A symmetric functor from $\tilde{\mathcal{C}}$ to $\tilde{\mathcal{C}}'$ is a tensor functor (F, J, φ) such that the diagram

$$\begin{array}{ccc}
F(X) \otimes' F(Y) & \xrightarrow{c'_{F(X) \otimes' F(Y)}} & F(Y) \otimes' F(X) \\
J_{X,Y} \downarrow & & J_{Y,X} \downarrow \\
F(X \otimes Y) & \xrightarrow{F(c_{X,Y})} & F(Y \otimes X)
\end{array} \tag{14}$$

is commutative for all $X, Y \in \text{Ob}(\mathcal{C})$.

Example 2.9 Let k be any field.

1. The categories Vec_k and $\text{Rep}_k(G)$ are symmetric with $c = \tau$.
2. The tensor category $\text{Supervec}_k := \text{Rep}_k(\mathbb{Z}_2)$ of super-vector spaces has a symmetric structure defined by $c_{X,Y}(x \otimes y) = (-1)^{|x||y|}(y \otimes x)$ where x, y are homogeneous elements and $|x| \in \{0, 1\}$ is the degree of x . Note that as *symmetric* categories Supervec_k and $\text{Rep}_k(\mathbb{Z}_2)$ are *not* equivalent.

In any symmetric rigid category, there is a natural notion of dimension, generalizing the ordinary dimension of an object in Vec_k .

Definition 2.10 Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$ be a symmetric rigid category and $X \in \text{Ob}(\mathcal{C})$. The categorical dimension $\dim_{\mathcal{C}}(X) \in \text{End}(\mathbf{1})$ of X is the morphism

$$\dim_{\mathcal{C}}(X) : \mathbf{1} \xrightarrow{ev_X} X \otimes X^* \xrightarrow{c_{X,X^*}} X^* \otimes X \xrightarrow{coev_X} \mathbf{1}. \tag{15}$$

Remark 2.11 Categorical dimensions are preserved by symmetric functors.

3 Deligne's theorem on Tannakian categories

Let k be an algebraically closed field. Let $\tilde{\mathcal{C}} := (\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$ be a k -linear abelian symmetric rigid category. An exact and faithful symmetric functor $F : \tilde{\mathcal{C}} \rightarrow \text{Vec}_k$ is called a *fiber functor* [DM].

Definition 3.1 *A k -linear abelian symmetric rigid category which admits a fiber functor is called Tannakian.*

Example 3.2 The category $\text{Rep}_k(G)$ (see Example 2.9) is Tannakian. In this situation the forgetful functor to Vec_k is a fiber functor. Note that $\text{End}(\mathbf{1}) = k$.

In fact, we have the following fundamental theorem of Deligne and Milne:

Theorem 3.3 (DM, Theorem 2.11) *Let $\tilde{\mathcal{C}} := (\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$ be a Tannakian category over k such that $\text{End}(\mathbf{1}) = k$, and let F be a fiber functor to Vec_k . Then $\tilde{\mathcal{C}}$ is equivalent, as a symmetric rigid category, to $\text{Rep}_k(G)$ where $G := \text{Aut}^\otimes(F)$ is an affine proalgebraic group.*

The following deep theorem of Deligne gives an intrinsic characterization of Tannakian categories.

Theorem 3.4 (De, Theorem 7.1) *Let $\tilde{\mathcal{C}} := (\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$ be a k -linear abelian symmetric rigid category over a field k of characteristic 0, with $\text{End}(\mathbf{1}) = k$. If $\dim_{\mathcal{C}}(X) \in \mathbb{Z}^+$ for all $X \in \text{Ob}(\mathcal{C})$, then $\tilde{\mathcal{C}}$ is Tannakian.*

As a corollary of Theorems 3.3 and 3.4 we have the following special case which we shall use later in the proof of our main theorem.

Corollary 3.5 *Let k be an algebraically closed field of characteristic 0, and $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$ a k -linear abelian symmetric rigid category with $\text{End}(\mathbf{1}) = k$, which is semisimple with finitely many irreducible objects. If categorical dimensions of objects are non-negative integers, then there exist a finite group G and an equivalence of symmetric rigid categories $F : \mathcal{C} \rightarrow \text{Rep}_k(G)$.*

Proof: Let G be the affine proalgebraic group whose existence is guaranteed by Theorems 3.3 and 3.4. The semisimplicity assumption, and the assumption that G has only finite number of irreducible representations imply that G is finite.

4 The category of finite-dimensional representations of a triangular Hopf algebra

In this section we relate (symmetric) tensor rigid categories and Hopf algebras. Let $(A, m, 1)$ be a finite-dimensional associative algebra with multiplication map m and unit element 1 over a field k , and let $\mathcal{C} := \text{Rep}_k(A)$ be the category of finite-dimensional left A -modules. Clearly, \mathcal{C} is a k -linear abelian category. In fact, the algebra A can be reconstructed from \mathcal{C} . Indeed, let Forget be the forgetful functor $\mathcal{C} \rightarrow \text{Vec}_k$, and let $\theta : A \rightarrow \text{End}(\text{Forget})$ be the map given by $\theta(a)_V := a|_V$, for all $a \in A$. Then we have the following:

Theorem 4.1 *The map θ defines an isomorphism of algebras between A and $\text{End}(\text{Forget})$.*

Proof: See e.g. [ES, Theorem 18.1].

Suppose further that A is a *Hopf algebra*. That is, there exist 3 additional structure maps

$$\Delta : A \rightarrow A \otimes A, \quad \varepsilon : A \rightarrow k \text{ and } S : A \rightarrow A$$

called the comultiplication, counit and antipode respectively, satisfying the following properties:

$$\Delta, \varepsilon \text{ are algebra homomorphisms, and } S \text{ is an algebra anti-isomorphism,} \quad (16)$$

$$(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta \text{ (i.e. } \Delta \text{ is coassociative),} \quad (17)$$

$$(\varepsilon \otimes id_A) \circ \Delta = (id_A \otimes \varepsilon) \circ \Delta = id_A \text{ and} \quad (18)$$

$$(S \otimes id_A) \circ \Delta = (id_A \otimes S) \circ \Delta = \varepsilon \cdot 1. \quad (19)$$

What does it say about the structure of \mathcal{C} ? First note that the field k becomes an object of \mathcal{C} thanks to ε being an algebra homomorphism:

$$a \cdot x = \varepsilon(a)x \text{ for any } a \in A \text{ and } x \in k. \quad (20)$$

Second, for any $V, W \in \text{Ob}(\mathcal{C})$, $V \otimes W$ becomes an object of \mathcal{C} thanks to Δ being an algebra homomorphism:

$$a \cdot (v \otimes w) = \Delta(a) \cdot (v \otimes w) \text{ for any } a \in A, v \in V \text{ and } w \in W. \quad (21)$$

Third, for any $V \in \text{Ob}(\mathcal{C})$, its linear dual V^* becomes an object of \mathcal{C} thanks to S being an algebra anti-homomorphism:

$$(a \cdot f)(v) = f(S(a) \cdot v) \text{ for any } a \in A, v \in V \text{ and } f \in V^*. \quad (22)$$

Moreover, let $V \in \text{Ob}(\mathcal{C})$ and fix dual bases $\{v_i\}$ and $\{f_i\}$ of V, V^* respectively. Then it is straightforward to verify that the k -linear maps

$$ev_V : V^* \otimes V \rightarrow k, \quad f \otimes v \mapsto f(v) \quad (23)$$

and

$$\text{coev}_V : k \rightarrow V \otimes V^* \text{ determined by } 1 \mapsto \sum_i v_i \otimes f_i \quad (24)$$

are in fact A -module maps. It is now straightforward to verify that (17)-(24) say precisely that $(\mathcal{C}, \otimes, k, a, l, r)$ is a tensor rigid category where a, l, r are the usual constraints (as in Vec_k). For instance, a is an A -module map thanks to (17), and l, r are A -module maps thanks to (18).

Suppose further that $(A, m, 1, \Delta, \varepsilon, S, R)$ is *triangular* [Dr]. That is, $R = \sum_i a_i \otimes b_i \in A \otimes A$ is invertible with $R^{-1} = R_{21}$ (R is unitary) and the following axioms hold:

$$(\Delta \otimes \text{id}_A)(R) = R_{13}R_{23}, (\text{id}_A \otimes \Delta)(R) = R_{13}R_{12} \quad (25)$$

and

$$\Delta^{\text{cop}}(a)R = R\Delta(a) \text{ for all } a \in A \quad (26)$$

(here $\Delta^{\text{cop}} = \tau \circ \Delta$ where $\tau : A \otimes A \rightarrow A \otimes A$ is the usual permutation map). Note that the two identities of (25) are equivalent since R is unitary. However, when R is not unitary (i.e. (A, R) is *quasitriangular* [Dr]) they are not equivalent. Following [R] we shall say that (A, R) is *minimal* when R , considered as an element in $\text{Hom}_k(A^*, A)$, defines an isomorphism.

What an extra structure does it put on \mathcal{C} ? Well, thanks to R we can define, for any $X, Y \in \text{Ob}(\mathcal{C})$, a k -linear map

$$c_{X,Y} : X \otimes Y \rightarrow Y \otimes X, x \otimes y \mapsto \tau(R \cdot (x \otimes y)). \quad (27)$$

This is in fact an A -module map thanks to (26). It is now straightforward to verify that the collection $c := \{c_{X,Y} | X, Y \in \text{Ob}(\mathcal{C})\}$ determines a symmetric structure on $(\mathcal{C}, \otimes, k, a, l, r)$. For instance, $c_{X,Y} \circ c_{Y,X} = \text{id}_{Y \otimes X}$ thanks to the unitarity of R , and (12) is satisfied thanks to (25).

In fact, the converse holds as well. Namely, given a symmetric rigid structure on \mathcal{C} such that the forgetful functor $F := \text{Forget} : \mathcal{C} \rightarrow \text{Vec}_k$ is exact, faithful and tensor, one can construct a triangular Hopf algebra structure on the algebra A (see e.g. [ES, Theorem 18.3]). Let us sketch this construction.

Recall from Theorem 4.1 that the algebras A and $\text{End}(F)$ are isomorphic. Similarly, one can show that the algebras $A \otimes A$ and $\text{End}(F^2)$ are isomorphic, where F^2 is the functor $\mathcal{C} \times \mathcal{C} \rightarrow \text{Vec}_k$, defined by $(V, W) \mapsto F(V) \otimes F(W)$. Now, using the tensor structure J on F , define the linear map

$$\Delta : \text{End}(F) \rightarrow \text{End}(F^2), \Delta(\eta)_{V,W} := J_{V,W}^{-1} \circ \eta_{V \otimes W} \circ J_{V,W}. \quad (28)$$

Also, define the linear maps

$$\varepsilon : \text{End}(F) \rightarrow \text{End}(F(\mathbf{1})) = k, \varepsilon(\eta) := \eta_{F(\mathbf{1})} \quad (29)$$

and

$$S : \text{End}(F) \rightarrow \text{End}(F), S(\eta)_V := (\eta_{V^*})^*. \quad (30)$$

Finally, set

$$R := \tau((c_{A,A}(1 \otimes 1)). \quad (31)$$

Then one can show that $(A, m, 1, \Delta, \varepsilon, S, R)$ is a triangular Hopf algebra.

Summarizing the above we state the following Tannaka-Krein type theorem.

Theorem 4.2 *The assignment described above defines a bijection between:*

1. *symmetric rigid structures on \mathcal{C} such that the forgetful functor $\text{Forget} : \mathcal{C} \rightarrow \text{Vec}_k$ is exact, faithful and tensor, modulo equivalence and isomorphism of tensor functors, and*
2. *triangular Hopf algebra structures on $(A, m, 1)$, modulo isomorphism.*

From now on we assume that A is a *triangular* Hopf algebra. Let us now determine the categorical dimensions in \mathcal{C} . Let

$$u := \sum_i S(b_i) a_i \quad (32)$$

be the *Drinfeld element* of (A, R) . Drinfeld showed [Dr] that u is invertible,

$$uxu^{-1} = S^2(x), \text{ for any } x \in A, \quad (33)$$

and

$$\Delta(u) = u \otimes u \text{ (i.e. } u \text{ is a grouplike element)}. \quad (34)$$

Lemma 4.3 *The categorical dimension $\dim_{\mathcal{C}}(X) \in k$ of an A -module X is given by $\text{tr}_X(u)$.*

Proof: By (15), $\dim_{\mathcal{C}}(X) = \sum_i (b_i \cdot f_i)(a_i \cdot x_i) = \sum_i f_i(S(b_i) a_i \cdot x_i) = \sum_i f_i(u \cdot x_i) = \text{tr}_X(u)$ as desired.

Suppose further that $(A, m, 1, \Delta, \varepsilon, S, R)$ is *semisimple* (i.e. A is a semisimple algebra).

Lemma 4.4 *The Drinfeld element u is central, and*

$$u = S(u). \quad (35)$$

Proof: By a fundamental result of Larson and Radford [LR], $S^2 = \text{id}_A$, and hence by (33), u is central. Now, we have $(S \otimes S)(R) = R$ [Dr], so $S(u) = \sum_i S(a_i) S^2(b_i) = \sum_i a_i S(b_i)$. This shows that $\text{tr}(u) = \text{tr}(S(u))$ in every irreducible representation of A . But u and $S(u)$ are central, so they act as scalars in this representation, which proves (35).

Lemma 4.5 *In particular,*

$$u^2 = 1. \quad (36)$$

Proof: By (34), $S(u) = u^{-1}$, hence the result follows from (35).

Let us demonstrate that it is always possible to replace R with a new R -matrix \tilde{R} so that the new Drinfeld element \tilde{u} equals 1. Indeed, for any irreducible representation V of A , define its parity, $p(V) \in \mathbb{Z}_2$, by $(-1)^{p(V)} = u|_V$. Define $\tilde{R} \in A \otimes A$ by the condition $\tilde{R}|_{V \otimes W} = (-1)^{p(V)p(W)} R|_{V \otimes W}$. Set

$$R_u := \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u). \quad (37)$$

Lemma 4.6 $\tilde{R} = RR_u$, and (A, RR_u) is semisimple triangular with Drinfeld element 1.

Proof: Straightforward.

This observation allows to reduce questions about semisimple triangular Hopf algebras over k to the case when the Drinfeld element is 1.

Remark 4.7 One should distinguish between the categorical dimensions of objects, defined in any braided rigid category, and their *quantum dimensions*, defined only in a ribbon category (see e.g. [Kas] for the definition of a ribbon category). In the diagrammatic language of [Kas] the quantum dimension corresponds to a loop without self-crossing, and the categorical dimension to a loop with one self-overcrossing. They may be different numbers for a particular irreducible object. For example, in the category of representations of a semisimple triangular Hopf algebra (A, R) , quantum dimensions (for an appropriate ribbon structure) are ordinary dimensions, while categorical dimensions are $u|_V \dim(V)$, where $u|_V$ is the scalar by which the Drinfeld element u acts on V , i.e. 1 or -1 .

5 Triangular semisimple Hopf algebras arising from twisting in group algebras of finite groups

Let $(A, m, 1, \Delta, \varepsilon, S)$ be a Hopf algebra over k . Recall [Dr] that a *twist* for A is an invertible element $J \in A \otimes A$ which satisfies

$$(\Delta \otimes id_A)(J)(J \otimes 1) = (id_A \otimes \Delta)(J)(1 \otimes J) \text{ and } (\varepsilon \otimes id_A)(J) = (id_A \otimes \varepsilon)(J) = 1. \quad (38)$$

Drinfeld noticed that given a twist J for A , one can define a new Hopf algebra structure $(A^J, m, 1, \Delta^J, \varepsilon, S^J)$ on the algebra $(A, m, 1)$ where the coproduct Δ^J is determined by

$$\Delta^J(x) = J^{-1} \Delta(x) J \quad (39)$$

for all $x \in A$, and the antipode S^J is determined by

$$S^J(x) = Q^{-1} S(x) Q \quad (40)$$

for all $x \in A$, where $Q := m \circ (S \otimes id_A)(J)$. If A is triangular with the universal R -matrix R , then so is A^J , with the universal R -matrix $R^J := J_{21}^{-1} R J$ (where $J_{21} := \tau(J)$).

If J is a twist for A and x is an invertible element of A such that $\varepsilon(x) = 1$, then

$$J^x := \Delta(x) J (x^{-1} \otimes x^{-1})$$

is also a twist for A . We will call the twists J and J^x *gauge equivalent*. Observe that the map $(A^J, R^J) \rightarrow (A^{J^x}, R^{J^x})$ determined by $a \mapsto xax^{-1}$ is an isomorphism of triangular Hopf algebras. Note that if J is a twist for A , then J^{-1} is a twist for A^J .

Example 5.1 Let G be a finite group. Then $(k[G], 1 \otimes 1)$ is semisimple triangular. Let $J \in k[G] \otimes k[G]$ be a twist. Then $(k[G]^J, J_{21}^{-1} J)$ is also semisimple triangular. When it is minimal triangular we shall say that J is a *minimal* twist.

Theorem 5.2 *The categories $\text{Rep}_k(A)$ and $\text{Rep}_k(A^J)$ are equivalent as symmetric rigid categories.*

Proof: Let $F : \text{Rep}_k(A) \rightarrow \text{Rep}_k(A^J)$ be the functor defined by $F(X) = X$. Let $\varphi = id_k$ and $J = \{J_{X,Y}\}$, where $J_{X,Y} : X \otimes Y \rightarrow X \otimes Y$ is defined by $J_{X,Y}(x \otimes y) = J \cdot (x \otimes y)$. Then it is straightforward to verify that (F, J, φ) is a symmetric functor.

The main result of this section is the following uniqueness theorem:

Theorem 5.3 *Let G, G' be finite groups, H, H' subgroups of G, G' respectively, and J, J' minimal twists for $k[H], k[H']$ respectively. Suppose that the triangular Hopf algebras $(k[G]^J, J_{21}^{-1} J), (k[G']^{J'} J_{21}'^{-1} J')$ are isomorphic. Then there exists a group isomorphism $\phi : G \rightarrow G'$ such that $\phi(H) = H'$, and $(\phi \otimes \phi)(J)$ is gauge equivalent to J' as twists for $k[H']$.*

The rest of the section is devoted to the proof of Theorem 5.3.

Lemma 5.4 *Let \mathcal{C} be the category of finite-dimensional representations over k of a finite group, and $F_1, F_2 : \mathcal{C} \rightarrow \text{Vec}_k$ be two fiber functors. Then F_1 is isomorphic to F_2 .*

Proof: This is a special case of [DM, Theorem 3.2]. This theorem states that the category of fiber functors from $\text{Rep}_k(G)$ to Vec_k , G an affine proalgebraic group, is equivalent to the category of G -torsors over k . But, k is algebraically closed, hence there exists only a unique G -torsor over k .

The following corollary of this lemma answers positively Movshev's question [Mov, Remark 1] whether any symmetric twist is trivial.

Corollary 5.5 *Let G be a finite group, and J be a symmetric twist for $k[G]$ (i.e. $J_{21} = J$). Then J is gauge equivalent to $1 \otimes 1$.*

Proof: Let $\mathcal{C} := \text{Rep}_k(G)$. We have two fiber functors $F_1, F_2 : \mathcal{C} \rightarrow \text{Vec}_k$ arising from the forgetful functor; namely, the trivial one and the one defined by J respectively. By Lemma 5.4, F_1, F_2 are isomorphic. Let $\eta : F_1 \rightarrow F_2$ be an isomorphism. By definition, $\eta = \{\eta_V : V \rightarrow V \mid V \in \text{Ob}(\mathcal{C})\}$ is a family of k -linear isomorphisms. By naturality, the diagram

$$\begin{array}{ccc} F_1(V) & \xrightarrow{\eta_V} & F_2(V) \\ F_1(f) \downarrow & & \downarrow F_2(f) \\ F_1(W) & \xrightarrow{\eta_W} & F_2(W) \end{array} \quad (41)$$

commutes for any two objects $V, W \in \text{Ob}(\mathcal{C})$ and morphism $f : V \rightarrow W$. In particular, for $V = W := k[G]$, the left regular representation, we get that $f \circ \eta_{k[G]} = \eta_{k[G]} \circ f$. Let $g \in k[G]$ and $f := r_g : k[G] \rightarrow k[G]$ be the right multiplication by g . Then, $r_g \circ \eta_{k[G]} = \eta_{k[G]} \circ r_g$, which is equivalent to saying that $\eta_{k[G]} : k[G] \rightarrow k[G]$ is an isomorphism of the *right* regular representation. Hence, $\eta_{k[G]}(a) = ax$ for all $a \in k[G]$, where $x := \eta_{k[G]}(1) \in k[G]$ is invertible. Moreover, setting $V := k[G]$, $W := k[G] \otimes k[G]$ and $f := \Delta$ in (41) yields that $(\Delta \circ \eta_{k[G]})(1) = (\eta_{k[G] \otimes k[G]} \circ \Delta)(1)$ which is equivalent to $\Delta(x) = \eta_{k[G] \otimes k[G]}(1 \otimes 1)$.

Now, by (9), the diagram

$$\begin{array}{ccc} F_1(k[G]) \otimes F_1(k[G]) & \xrightarrow{1 \otimes 1} & F_1(k[G] \otimes k[G]) \\ \eta_{k[G]} \otimes \eta_{k[G]} \downarrow & & \downarrow \eta_{k[G] \otimes k[G]} \\ F_2(k[G]) \otimes F_2(k[G]) & \xrightarrow{J} & F_2(k[G] \otimes k[G]) \end{array} \quad (42)$$

commutes. In particular, $J(\eta_{k[G]} \otimes \eta_{k[G]})(1 \otimes 1) = \eta_{k[G] \otimes k[G]}(1 \otimes 1)$. Hence, $J(x \otimes x) = \Delta(x)$ as desired.

Remark 5.6 Here is another proof of Corollary 5.5 which does not use Lemma 5.4 but uses the results of [Mov] (see also [G2] for details). Consider the G -coalgebra $A_J := k[G]$ with coproduct Δ_J determined by $\Delta_J(x) = (x \otimes x)J$, $x \in G$, and its dual algebra $(A_J)^*$. According to [Mov], this algebra is semisimple, G acts transitively on its simple ideals, and $(A_J)^*$, along with the action of G , completely determines J up to gauge transformations. Clearly, since J is symmetric, this algebra is commutative. So, it is isomorphic, as a G -algebra, to the algebra of functions on a set X on which G acts simply transitively. Corollary 5.5 now follows from the fact that such a G -set is unique up to an isomorphism (the group G itself with G acting by left multiplication). Note that this proof also uses the uniqueness of G -torsors (the set X is a G -torsor).

Lemma 5.7 *Let G, G' be finite groups, J, J' twists for $k[G], k[G']$ respectively, and suppose that the triangular Hopf algebras $(k[G]^J, J_{21}^{-1}J)$, $(k[G']^{J'}, J'_{21}{}^{-1}J')$ are isomorphic. Then there exists a group isomorphism $\phi : G \rightarrow G'$ such that $(\phi \otimes \phi)(J)$ is gauge equivalent to J' .*

Proof: Let $f : k[G]^J \rightarrow k[G']^{J'}$ be an isomorphism of triangular Hopf algebras. Then f defines an isomorphism of triangular Hopf algebras from $k[G]$ to $k[G']^{J'(f \otimes f)(J)^{-1}}$. This implies that the element $J'(f \otimes f)(J)^{-1}$ is a symmetric twist for $k[G']$. Thus, for some invertible $x \in k[G']$ one has $J'(f \otimes f)(J)^{-1} = \Delta(x)(x^{-1} \otimes x^{-1})$. Let $\phi := \text{Ad}(x^{-1}) \circ f : k[G] \rightarrow k[G']$. It is obvious that ϕ is a Hopf algebra isomorphism, so it comes from a group isomorphism $\phi : G \rightarrow G'$. We have $(\phi \otimes \phi)(J) = \Delta(x)^{-1} J'(x \otimes x)$, as desired.

We can now prove Theorem 5.3. By Lemma 5.7, it is sufficient to assume that $G' = G$, and that J is gauge equivalent to J' as twists for $k[G]$, and it is enough to show that there exists an element $a \in G$ such that $aHa^{-1} = H'$ and $(a \otimes a)J(a^{-1} \otimes a^{-1})$ is gauge equivalent to J' as twists for $k[H']$.

So let $x \in k[G]$ be the invertible element such that $\Delta(x)J(x^{-1} \otimes x^{-1}) = J'$. In particular, this implies that $(x \otimes x)R(x^{-1} \otimes x^{-1}) = R'$, where R, R' are the R -matrices corresponding to J, J' respectively. By the minimality of J, J' , we have $xk[H]x^{-1} = k[H']$. Thus,

$$J_0 := \Delta(x)(x^{-1} \otimes x^{-1}) = J'(x \otimes x)J^{-1}(x^{-1} \otimes x^{-1}) \in k[H']^{\otimes 2}.$$

It is obvious that J_0 is a symmetric twist for $k[H']$, so by Corollary 5.5, it is gauge equivalent to $1 \otimes 1$. Thus, $x = x_0 a$, for some invertible $x_0 \in k[H']$, and $a \in G$. It is clear that $aHa^{-1} = H'$, and $\Delta(x_0^{-1})J'(x_0 \otimes x_0) = (a \otimes a)J(a^{-1} \otimes a^{-1})$. This concludes the proof of Theorem 5.3.

6 The main theorem

We can now state and prove our main result:

Theorem 6.1 *Let (A, R) be a semisimple triangular Hopf algebra over an algebraically closed field k of characteristic 0, with Drinfeld element u , and set $\tilde{R} := RR_u$. Then there exist a finite group G and a twist $J \in k[G] \otimes k[G]$ such that (A, \tilde{R}) and $(k[G]^J, J_{21}^{-1}J)$ are isomorphic as triangular Hopf algebras. That is, there exists an algebra isomorphism $\phi : k[G] \rightarrow A$ satisfying $\Delta(\phi(a)) = (\phi \otimes \phi)(J^{-1}\Delta(a)J)$, $\varepsilon(\phi(a)) = \varepsilon(a)$ for all $a \in k[G]$, and $(\phi \otimes \phi)(J_{21}^{-1}J) = \tilde{R}$.*

Proof: Let \mathcal{C} be the category of finite-dimensional representations over k of (A, \tilde{R}) . This is a semisimple abelian k -linear category with finitely many irreducible objects, which has a structure of a symmetric rigid category (see Section 4). In this case, the categorical dimension $\dim_{\mathcal{C}}(V)$ of $V \in \text{Ob}(\mathcal{C})$ is equal to $\text{tr}_V(\tilde{u})$. Since the Drinfeld element \tilde{u} of (A, \tilde{R}) is 1, it follows that it is equal to the ordinary dimension of V as a vector space. In particular, all categorical dimensions are non-negative integers. In this situation we can apply Theorem 3.5.

Let G, F be the finite group and functor corresponding to our category \mathcal{C} . In particular, F preserves categorical dimensions of objects, and hence their ordinary dimensions. Thus, we may identify $V, F(V)$ as vector spaces functorially for all $V \in \text{Ob}(\mathcal{C})$.

Lemma 6.2 *There exists an algebra isomorphism $\phi : k[G] \rightarrow A$ such that for all object $V \in \text{Ob}(\mathcal{C})$, the G -module structure on $F(V)$ is given via pull back along ϕ .*

Proof: Let $\{V_i | 0 \leq i \leq m\}$ be the set of all the isomorphism classes of irreducible representations of A . Since F is an equivalence of tensor rigid categories $\{F(V_i) = V_i | 0 \leq i \leq m\}$ is the set of all the isomorphism classes of irreducible representations of G . Since A and $k[G]$ are semisimple algebras we can fix algebra isomorphisms $k[G] \rightarrow \bigoplus_{i=0}^m \text{End}(V_i)$ and $\bigoplus_{i=0}^m \text{End}(V_i) \rightarrow A$. This determines an isomorphism of algebras $\phi : k[G] \rightarrow A$ (of course, this isomorphism is not unique). Now, by the construction of ϕ , the vector space $F(V_i) = V_i$ is a G -module via pull back along ϕ .

By Definition 2.3, there exists a family of natural G -module isomorphisms $J_{V,W} : F(V) \otimes F(W) \rightarrow F(V \otimes W)$ indexed by all couples $(V, W) \in \text{Ob}(\mathcal{C} \times \mathcal{C})$. Consider $J_{A,A} : F(A) \otimes F(A) \rightarrow F(A \otimes A)$, and set

$$\tilde{J} := J_{A,A}(1 \otimes 1) \in A \otimes A.$$

Lemma 6.3 *For all $V, W \in \text{Ob}(\mathcal{C})$ and $v \in V, w \in W$, $J_{V,W}(v \otimes w) = \tilde{J} \cdot (v \otimes w)$.*

Proof: Consider the A -module maps $\bar{v} : A \rightarrow V$ and $\bar{w} : A \rightarrow W$ determined by $a \mapsto a \cdot v$ and $a \mapsto a \cdot w$ for all $a \in A$ respectively. By naturality, the diagram

$$\begin{array}{ccc} F(A) \otimes F(A) & \xrightarrow{J_{A,A}} & F(A \otimes A) \\ F(\bar{v}) \otimes F(\bar{w}) \downarrow & & F(\bar{v} \otimes \bar{w}) \downarrow \\ F(V) \otimes F(W) & \xrightarrow{J_{V,W}} & F(V \otimes W) \end{array} \quad (43)$$

commutes. In particular, $(J_{V,W} \circ (F(\bar{v}) \otimes F(\bar{w})))(1 \otimes 1) = (F(\bar{v} \otimes \bar{w}) \circ J_{A,A})(1 \otimes 1)$, which is equivalent to $J_{V,W}(v \otimes w) = J_{A,A}(1 \otimes 1) \cdot (v \otimes w)$ as desired.

In particular, the isomorphism $J_{A,A} : F(A) \otimes F(A) \rightarrow F(A \otimes A)$ is determined by $x \otimes y \mapsto \tilde{J}(x \otimes y)$. Since $1 \otimes 1$ is in its image, it follows that \tilde{J} is invertible. Hence $(\phi^{-1} \otimes \phi^{-1})(\tilde{J}) \in k[G] \otimes k[G]$ is invertible as well. Set

$$J := (\phi^{-1} \otimes \phi^{-1})(\tilde{J})^{-1} \in k[G] \otimes k[G]. \quad (44)$$

Lemma 6.4 *For all $a \in k[G]$, $\Delta(\phi(a)) = (\phi \otimes \phi)(J^{-1} \Delta(a) J)$.*

Proof: Since the map $J_{A,A} : F(A) \otimes F(A) \rightarrow F(A \otimes A)$ is an isomorphism of G -modules, we have that $J_{A,A}(a \cdot (1 \otimes 1)) = a \cdot J_{A,A}(1 \otimes 1)$. By Lemma 6.3, this is equivalent to

$$\tilde{J}((\phi \otimes \phi)(\Delta(a))) = \Delta(\phi(a)) \tilde{J}.$$

The claim follows now after replacing \tilde{J} by $(\phi \otimes \phi)(J^{-1})$.

Lemma 6.5 For all $a \in k[G]$, $\varepsilon(\phi(a)) = \varepsilon(a)$.

Proof: We first show that $(\varepsilon \otimes id_A)(\tilde{J}) = (id_A \otimes \varepsilon)(\tilde{J}) = 1$. Let r denote the right unit constraints (we use the same notation for both categories for convenience). Then by (8), we have that the diagram

$$\begin{array}{ccc} F(A) \otimes k & \xrightarrow{r_{F(A)}} & F(A) \\ id \otimes id \downarrow & & \uparrow F(r_A) \\ F(A) \otimes F(k) & \xrightarrow{J_{A,k}} & F(A \otimes k) \end{array} \quad (45)$$

commutes. In particular, $(F(r_A) \circ J_{A,k} \circ (id \otimes id))(1 \otimes 1) = r_{F(A)}(1 \otimes 1)$ which implies that $(\varepsilon \otimes id_A)(\tilde{J}) = 1$. Similarly, $(id_A \otimes \varepsilon)(\tilde{J}) = 1$.

Now, since the map $J_{A,k} : F(A) \otimes k \rightarrow F(A \otimes k)$ is an isomorphism of G -modules, we have that $J_{A,k}(a \cdot (1 \otimes 1)) = a \cdot J_{A,k}(1 \otimes 1)$. By Lemma 6.3, this is equivalent to $\tilde{J} \cdot ((\phi \otimes \phi)(\Delta_0(a)) \cdot (1 \otimes 1)) = \Delta(\phi(a)) \cdot (\tilde{J} \cdot (1 \otimes 1))$. Write $\tilde{J} = \sum_i x_i \otimes y_i$. Then the last equation implies that $\sum_i x_i \phi(a_1) \varepsilon(y_i) \varepsilon(\phi(a_2)) = \sum_i \phi(a)_1 x_i \varepsilon(\phi(a)_2) \varepsilon(y_i)$ which in turn (since ϕ is an isomorphism) implies that $\sum a_1 \varepsilon(\phi(a_2)) = a$, and the result follows.

Lemma 6.6 J is a twist for $k[G]$.

Proof: Let a denote the associativity constraints in the categories \mathcal{C} and $\text{Rep}_k(G)$ (we use the same notation for both categories for convenience). By (2), the diagram

$$\begin{array}{ccc} (F(A) \otimes F(A)) \otimes F(A) & \xrightarrow{a_{F(A), F(A), F(A)}} & F(A) \otimes (F(A) \otimes F(A)) \\ J_{A,A} \otimes id_{F(A)} \downarrow & & id_{F(A)} \otimes J_{A,A} \downarrow \\ F(A \otimes A) \otimes F(A) & & F(A) \otimes F(A \otimes A) \\ J_{A \otimes A, A} \downarrow & & J_{A, A \otimes A} \downarrow \\ F((A \otimes A) \otimes A) & \xrightarrow{F(a_{A,A,A})} & F(A \otimes (A \otimes A)) \end{array} \quad (46)$$

commutes. In particular,

$$\begin{aligned} & \left(F(a_{A,A,A}) \circ J_{A \otimes A, A} \circ (J_{A,A} \otimes id_{F(A)}) \right) (1 \otimes 1 \otimes 1) \\ &= \left(J_{A, A \otimes A} \circ (id_{F(A)} \otimes J_{A,A}) \circ a_{F(A), F(A), F(A)} \right) (1 \otimes 1 \otimes 1), \end{aligned}$$

which is equivalent to $(\Delta \otimes id_A)(\tilde{J})\tilde{J}_{12} = (id_A \otimes \Delta)(\tilde{J})\tilde{J}_{23}$. Write $J^{-1} = \sum_i x_i \otimes y_i$. Substitute $(\phi \otimes \phi)(J)^{-1}$ for \tilde{J} in the last equation, and use Lemma 6.4 to get

$$(\phi \otimes \phi \otimes \phi) \left((J^{-1} \Delta_0(x_i) J \otimes y_i) (J^{-1} \otimes 1) \right) = (\phi \otimes \phi \otimes \phi) \left((x_i \otimes J^{-1} \Delta_0(y_i) J) (1 \otimes J^{-1}) \right).$$

Since ϕ is an isomorphism, this is equivalent to saying that J satisfies the first part of (38).

Now, we already showed in the proof of Lemma 6.5 that

$$(\varepsilon \otimes id_A)(\tilde{J}) = (id_A \otimes \varepsilon)(\tilde{J}) = 1.$$

Thus, the second part of (38) follows from Lemma 6.4 after replacing \tilde{J} with $(\phi \otimes \phi)(J^{-1})$.

By Lemmas 6.2-6.6, $(k[G]^J, J_{21}^{-1}J)$ is a triangular semisimple Hopf algebra, and the map $\phi : k[G]^J \rightarrow A$ is an isomorphism of Hopf algebras. Finally, let c denote the commutativity constraints in the categories \mathcal{C} and $\text{Rep}_k(G)$ (again, we use the same notation for both categories for convenience). Recall that F is in particular a symmetric functor.

Lemma 6.7 $(\phi \otimes \phi)(J_{21}^{-1}J) = \tilde{R}$.

Proof: By Definition 2.8, the diagram

$$\begin{array}{ccc} F(A) \otimes F(A) & \xrightarrow{J_{A,A}} & F(A \otimes A) \\ c_{F(A), F(A)} \downarrow & & \downarrow F(c_{A,A}) \\ F(A) \otimes F(A) & \xrightarrow{J_{A,A}} & F(A \otimes A) \end{array} \quad (47)$$

commutes. In particular, $(F(c_{A,A}) \circ J_{A,A})(1 \otimes 1) = (J_{A,A} \circ c_{F(A), F(A)})(1 \otimes 1)$. Therefore, $\tilde{J} = \tilde{R}_{21} \tilde{J}_{21}$, which is equivalent to the desired result.

This completes the proof of the theorem.

Remark 6.8 As seen from Remark 4.7, if $u \neq 1$, then the category of representations of (A, R) is equivalent to the category of representations of some group as a rigid tensor category but *not* as a symmetric category. This was the reason for passing from R to \tilde{R} . It is easy to see that as a symmetric rigid category, the category of representations of (A, R) is equivalent to the category of representations of G on super-vector spaces, such that u acts by 1 on the even part and as -1 on the odd part. For example, if $A := k\mathbb{Z}_2$ with u as the generator of \mathbb{Z}_2 , and $R := R_u$, then the category of representations is just the category of super-vector spaces.

Corollary 6.9 *Let (A, R) be a semisimple triangular Hopf algebra over k with Drinfeld element 1. Then there exist finite groups $H \subseteq G$, and a minimal twist $J \in k[H] \otimes k[H]$ such that $(A, R) \cong (k[G]^J, J_{21}^{-1}J)$ as triangular Hopf algebras. Furthermore, the data (G, H, J) is unique up to isomorphism of groups and gauge equivalence of twists.*

Proof: Let (A_R, R) be the minimal triangular sub Hopf algebra of (A, R) . By Theorem 6.1, there exist a finite group H and a minimal twist J for $k[H]$ such that $(A_R, R) \cong (k[H]^J, J_{21}^{-1}J)$ as triangular Hopf algebras. We may as well assume that $(A_R, R) = (k[H]^J, J_{21}^{-1}J)$. Let $f : (k[H]^J, J_{21}^{-1}J) \rightarrow (A, R)$ be the inclusion map. Then $f : (k[H], 1 \otimes 1) \rightarrow (A^{J^{-1}}, J_{21}RJ^{-1})$ is an injective morphism of triangular Hopf algebras as well. In particular, $J_{21}RJ^{-1} = 1 \otimes 1$ which is equivalent to $R = J_{21}^{-1}J$. Moreover, since $(A^{J^{-1}}, 1 \otimes 1)$ is triangular, $A^{J^{-1}}$ is cocommutative. Therefore there exists a finite group G such that $A^{J^{-1}} = k[G]$. Hence, $(A, R) = (k[G]^J, J_{21}^{-1}J)$. Since $k[H]$ is a sub Hopf algebra of $k[G]$, H is a subgroup of G . Finally, the uniqueness follows from Theorem 5.3.

7 Concluding remarks

We conclude the paper with some related remarks and references. A full and detailed discussion of remarks 1-5 can be found in [G2].

Remark 7.1 1. The analogue of Theorem 6.1 in positive characteristic was obtained in [EG4]. In this situation one has to further assume that A is also cosemisimple (i.e. its dual Hopf algebra A^* is semisimple). The result that any semisimple cosemisimple triangular Hopf algebra over any algebraically closed field k is obtained from $k[G]$ for some finite group G , by twisting its usual comultiplication, follows then using the lifting functor from positive characteristic to characteristic 0 [EG5], the classification of twists for group algebras in characteristic 0 [EG4], and Theorem 6.1.

2. In fact much more can be said on the structure of semisimple cosemisimple triangular Hopf algebras over any algebraically closed field k . A complete classification of these objects was obtained, using Theorem 6.1 and the theory of Movshev [Mov], in [EG4]. It was proved there that the set of isomorphism classes of semisimple cosemisimple triangular Hopf algebras of dimension N over k is in bijection with the set of isomorphism classes of quadruples (G, H, V, u) where G is a finite group of order N , H is a subgroup of G , V is an irreducible projective representation of H of dimension $|H|^{1/2}$ and $u \in Z(G)$ is of order ≤ 2 . In particular, it follows that H is a quotient of a central type group and hence solvable by a result of Howlett and Isaacs [HI] (it is interesting to note that this result was proved using the classification of finite simple groups). In [EG2] we proved that any finite group with a bijective 1-cocycle to an abelian group gives rise to such a group, and hence to a minimal triangular semisimple Hopf algebra. As a corollary to the classification theorem we were able to prove that any semisimple cosemisimple triangular Hopf algebra has a non-trivial grouplike element [EG4]. In [EGGS] it was shown that this is false for semisimple quasitriangular Hopf algebras. A counterexample was constructed there using Mathieu simple group M_{24} .

3. A famous conjecture of Kaplansky from 1975 [Kap] states that the dimension of an irreducible representation of a semisimple Hopf algebra over k divides the dimension of A . This conjecture was proved, when A is quasitriangular, in [EG1] using the theory of modular categories. In the triangular case the conjecture also follows from Theorem 6.1 by a famous theorem of Frobenius. However, the conjecture does not follow easily for the duals of semisimple triangular Hopf algebras (semisimple *cotriangular* Hopf algebras). In [EG3], we used Theorem 6.1 to describe the representation theory of semisimple cotriangular Hopf algebras, and in particular proved that the conjecture holds for them as well.

4. There exist finite-dimensional triangular Hopf algebras which are *not* semisimple, so in particular are not obtained by twisting of group algebras. The simplest example is Sweedler's 4-dimensional Hopf algebra A [S]. It is generated as an algebra by a, x , subject to the relations $a^2 = 1$, $x^2 = 0$ and $ax = -xa$. Its coalgebra structure is determined by $\Delta(a) = a \otimes a$ and $\Delta(x) = x \otimes 1 + a \otimes x$. This Hopf algebra is *pointed* but not semisimple since it has only two non-isomorphic irreducible representations, both of dimension 1. However, A admits infinitely many minimal triangular structures and one nonminimal triangular structure (i.e.

R_a). See [G1] for a study of pointed non-semisimple finite-dimensional triangular Hopf algebras. In particular, this paper gives a classification of minimal triangular pointed Hopf algebras. The crucial property of such Hopf algebras is that they are also *basic*; i.e. all their irreducible representations are 1-dimensional. This implies that they share the *Chevalley property* with semisimple Hopf algebras; namely, their radical is a Hopf ideal and so their semisimple part is a Hopf algebra itself (see [AEG]). In representation-theoretic terms this means that the tensor product of any two irreducible representations is completely reducible. The fact that all known examples of finite-dimensional triangular Hopf algebras have the Chevalley property was the motivation to study such Hopf algebras in [AEG] and [EG8]. In [AEG] Theorem 6.1 was used to prove that a finite-dimensional triangular Hopf algebra has the Chevalley property if and only if it is obtained from a super-group algebra of a finite super-group after twisting its comultiplication. In [EG8] the main result of [EG4] (see 2) was generalized and it was proved that the isomorphism classes of finite-dimensional triangular Hopf algebras with the Chevalley property are in bijection with the isomorphism classes of septuples (G, W, H, Y, B, V, u) where G is a finite group, W is a finite-dimensional representation of G , H is a subgroup of G , Y is an H -invariant subspace of W , B is an H -invariant nondegenerate element in $S^2 Y$, V is an irreducible projective representation of H of dimension $|H|^{1/2}$ and $u \in Z(G)$ is of order ≤ 2 acting by -1 on W .

5. A natural question arising from Theorem 6.1 is what happens in the infinite-dimensional case. Suppose (A, R) is an infinite-dimensional cotriangular Hopf algebra over k . This is equivalent to saying that the category $\text{Corep}_k(A)$ of all the corepresentations of A is symmetric rigid. We can thus ask for a characterization of such Hopf algebras which are obtained by twisting (this time the usual *multiplication* of) the function algebra on some affine proalgebraic group. An answer to this question is given in [EG6]. It turns out that a necessary and sufficient condition for this to be the case is that A is pseudoinvolutive (i.e. $\text{tr}(S^2|_C) = \dim(C)$ for any finite dimensional subcoalgebra of A). In the finite-dimensional case, pseudoinvolutivity is equivalent to involutivity ($S^2 = \text{id}_A$), which in turn is equivalent to semisimplicity by a result of Larson and Radford. Thus this result generalizes Theorem 6.1. The proof also uses Deligne's theorem (Theorem 3.4). The hard direction is to prove that pseudoinvolutivity implies that the symmetry of $\text{Corep}_k(A)$ can be modified so that the categorical dimensions of objects will become non-negative integers.

6. As we have indicated above, if two finite groups have the same symmetric tensor categories of representations over \mathbf{C} , then they are isomorphic. This raises the following natural question: When do two finite groups G_1, G_2 have the same tensor categories of representations over \mathbf{C} (without regard for the commutativity constraint)? Two groups with such property are called *isocategorical* [EG9]. In [EG9] the theory of triangular semisimple Hopf algebras was used to completely answer this question. Namely, a classification of groups isocategorical to a given group G was given. In particular, it was showed that if G has no nontrivial normal abelian subgroups of order 2^{2m} then any group isocategorical to G must actually be isomorphic to G . On the other hand, an example of two groups which are isocategorical but not isomorphic was given: The affine symplectic group of a vector space over the field of two elements, and an appropriate "affine pseudosymplectic group" introduced by R. Griess (containing the "pseudosymplectic group" of A. Weil). Also the

notion of isocategorical groups was applied to answer the question: When are two triangular semisimple Hopf algebras isomorphic as Hopf algebras?

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